A noncommutative version of the nonlinear Schrödinger equation

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Abstract

We apply a (Moyal) deformation quantization to a bicomplex associated with the classical nonlinear Schrödinger equation. This induces a deformation of the latter equation to noncommutative space-time while preserving the existence of an infinite set of conserved quantities.

1 Introduction

Partly motivated by recent results on the appearance of field theories on noncommutative space-times in certain limits of string, D-brane and M theory [1], there has been a revival of (Moyal) deformation quantization [2] and increasing interest in models on noncommutative spaces (see [3], for example).

In this work we apply deformation quantization to a classical integrable model[1], the nonlinear Schrödinger equation. More precisely, we apply it to the two space-time coordinates (see also [6]) and generalize a bicomplex associated with the classical nonlinear partial differential equation to the resulting "quantized space-time". The deformed bicomplex equations are then equivalent to a deformed nonlinear Schrödinger equation which still possesses an infinite set of conservation laws. It is this latter characteristic property of soliton equations which distinguishes the resulting "quantized equation" and turns it into an interesting model.

Section 2 recalls some bicomplex formalism which has been developed in [7]. In section 3 we treat the nonlinear Schrödinger equation in this framework. Section 4 deals with the

[1]Deformation quantization in the context of integrable models has been considered in [4, 5, 6], for example.

corresponding noncommutative extension and in section 5 we show that the single soliton solution of the classical equation is also a solution of the noncommutative version. Section 6 deals with perturbative properties of this equation and section 7 contains some conclusions.

2 Bicomplexes and associated linear equation

A *bicomplex* is an $\mathbb{N}_0$-graded linear space (over $\mathbb{R}$ or $\mathbb{C}$) $M = \bigoplus_{r\geq 0} M^r$ together with two linear maps $\mathrm{d}, \delta : M^r \to M^{r+1}$ satisfying

$$\mathrm{d}^2 = 0\,, \qquad \delta^2 = 0\,, \qquad \mathrm{d}\,\delta + \delta\,\mathrm{d} = 0\,. \tag{2.1}$$

Let us assume that there is a (nonvanishing) $\chi^{(0)} \in M^0$ with $\mathrm{d}J^{(0)} = 0$ where $J^{(0)} = \delta\chi^{(0)}$. Let us define $J^{(1)} = \mathrm{d}\chi^{(0)}$. Then $\delta J^{(1)} = -\mathrm{d}\delta\chi^{(0)} = 0$. Hence $J^{(1)}$ is δ-closed. If $J^{(1)}$ is δ-exact, then $J^{(1)} = \delta\chi^{(1)}$ with some $\chi^{(1)} \in M^0$. Next we define $J^{(2)} = \mathrm{d}\chi^{(1)}$. Then $\delta J^{(2)} = -\mathrm{d}\delta\chi^{(1)} = -\mathrm{d}J^{(1)} = -\mathrm{d}^2\chi^{(0)} = 0$. If the δ-closed element $J^{(2)}$ is δ-exact, then $J^{(2)} = \delta\chi^{(2)}$ with some $\chi^{(2)} \in M^0$. This can be iterated further and leads to a (possibly infinite) chain of elements $J^{(m)}$ of M^1 and $\chi^{(m)} \in M^0$ satisfying

$$J^{(m+1)} = \mathrm{d}\chi^{(m)} = \delta\chi^{(m+1)}\,. \tag{2.2}$$

More precisely, the above iteration continues from the mth to the $(m+1)$th level as long as $\delta J^{(m)} = 0$ implies $J^{(m)} = \delta\chi^{(m)}$ with an element $\chi^{(m)} \in M^0$. Of course, there is no obstruction to the iteration if $H^1_\delta(M)$ is trivial, i.e., when all δ-closed elements of M^1 are δ-exact. In general, the latter condition is too strong, however. Introducing

$$\chi = \sum_{m\geq 0} \lambda^m\,\chi^{(m)} \tag{2.3}$$

as a (formal) power series in a parameter λ, the essential steps of the above iteration procedure are summarized in

$$\delta(\chi - \chi^{(0)}) = \lambda\,\mathrm{d}\,\chi \tag{2.4}$$

which we call the *linear equation* associated with the bicomplex.

Given a bicomplex, we may start directly with the linear equation (2.4). Let us assume that it admits a (non-trivial) solution χ as a (formal) power series (2.3) in λ. The linear equation then leads to

$$\delta\chi^{(m)} = \mathrm{d}\chi^{(m-1)}\,, \quad m = 1, \ldots, \infty\,. \tag{2.5}$$

As a consequence, the $J^{(m+1)} = \mathrm{d}\chi^{(m)}$ $(m = 0, \ldots, \infty)$ are δ-exact. Even if the cohomology $H^1_\delta(M)$ is *not* trivial, solvability of the linear equation ensures that the δ-closed $J^{(m)}$ appearing in the iteration are δ-exact.

For some integrable models the "generalized conserved currents" $J^{(m)}$ of an associated bicomplex are directly related to conserved densities [7]. In other cases, like the nonlinear Schrödinger equation treated in the following section, the relation is less direct.

3 Bicomplex formulation of the classical nonlinear Schrödinger equation

We choose the bicomplex space as $M = M^0 \otimes \Lambda$ where $M^0 = C^\infty(\mathbb{R}^2, \mathbb{C}^2)$ denotes the set of smooth maps $\phi : \mathbb{R}^2 \to \mathbb{C}^2$ and $\Lambda = \bigoplus_{r=0}^2 \Lambda^r$ is the exterior algebra of a 2-dimensional complex vector space with basis τ, ξ of Λ^1 (so that $\tau^2 = \xi^2 = \tau\,\xi + \xi\,\tau = 0$). It is then sufficient to define bicomplex maps d and δ on M^0 since by linearity and $\mathrm{d}(\phi_1\,\tau + \phi_2\,\xi) = (\mathrm{d}\phi_1)\,\tau + (\mathrm{d}\phi_2)\,\xi$ (and correspondingly for δ) they extend to the whole of M. Let

$$\mathrm{d}\phi = (\phi_t - V\,\phi)\,\tau + (\phi_x - U\,\phi)\,\xi \tag{3.1}$$

$$\delta\phi = \phi_x\,\tau + \frac{1}{2i}\,(I - \sigma_3)\,\phi\,\xi \tag{3.2}$$

where I is the 2×2 unit matrix, $\sigma_3 = \mathrm{diag}(1,-1)$, and

$$U = \begin{pmatrix} 0 & -\bar{\psi} \\ \psi & 0 \end{pmatrix}\,, \quad V = i\begin{pmatrix} -|\psi|^2 & \bar{\psi}_x \\ \psi_x & |\psi|^2 \end{pmatrix} \tag{3.3}$$

with a complex function ψ with complex conjugate $\bar{\psi}$. Then $\delta^2 = 0$ and $\mathrm{d}\delta + \delta\mathrm{d} = 0$ are identically satisfied. Furthermore, $\mathrm{d}^2 = 0$ holds iff $U_t - V_x + [U,V] = 0$ which is equivalent to the nonlinear Schrödinger equation

$$i\,\psi_t = -\psi_{xx} - 2\,|\psi|^2\,\psi\,. \tag{3.4}$$

The equation $\delta J = 0$ for an element $J = \phi_1\,\tau + \phi_2\,\xi \in M^1$ means $\phi_{2,x} = -(i/2)(I-\sigma_3)\phi_1$. Let $\Phi \in M^0$ such that $\Phi_x = \phi_1$. Then $\phi_2 = -(i/2)(I-\sigma_3)\Phi + C$ with an element $C(t) \in M^0$ and we find $J = \Phi_x\,\tau - [(i/2)(I-\sigma_3)\Phi - C]\,\xi = \delta\Phi + C\,\xi$. The cohomology of δ is not trivial since an element $(c(t),0)^T\xi$ is δ-closed, but obviously not δ-exact.[2]

Choosing $\chi^{(0)}$ such that[3] $\delta\chi^{(0)} = 0$, the linear equation associated with the bicomplex becomes

$$\chi_x = \lambda\,(\chi_t - V\chi)\,, \qquad (I-\sigma_3)\chi = 2\,i\,\lambda\,(\chi_x - U\chi)\,. \tag{3.5}$$

In terms of $\chi = (\alpha,\beta)^T$ this system of equations takes the form

$$\alpha_x = \lambda\,\{\alpha_t + i\,|\psi|^2\alpha - i\,\bar{\psi}_x\beta\} \tag{3.6}$$

$$\beta_x = \lambda\,\{\beta_t - i\,\psi_x\,\alpha - i\,|\psi|^2\beta\} \tag{3.7}$$

$$\alpha_x = -\bar{\psi}\,\beta \tag{3.8}$$

$$\beta = i\,\lambda\,(\beta_x - \psi\,\alpha)\,. \tag{3.9}$$

The third equation can be used to eliminate β in the other equations (assuming $\psi \neq 0$). We obtain

$$\alpha_x = \lambda\,\{\alpha_t + i\,|\psi|^2\alpha + i\,(\bar{\psi}_x/\bar{\psi})\,\alpha_x\} \tag{3.10}$$

$$\alpha_x = i\,\lambda\,(\alpha_{xx} - (\bar{\psi}_x/\bar{\psi})\alpha_x + |\psi|^2\alpha)\,. \tag{3.11}$$

[2] ϕ^T denotes the transpose of ϕ.

[3] The general solution of $\delta\chi^{(0)} = 0$ is $(c(t),0)^T$.

Conversely, if these two equations hold and if ψ satisfies the nonlinear Schrödinger equation, then the linear equation is satisfied (where β is defined via (3.8)). Moreover, the above two equations for α are compatible, i.e., the equation obtained by differentiating the second equation with respect to t is identically satisfied as a consequence of these equations together with the nonlinear Schrödinger equation.

Now we choose $\chi^{(0)} = (1,0)^T$. In terms of γ determined by

$$\alpha = e^{i\lambda\gamma} \,, \qquad \gamma = \sum_{m\geq 0} \lambda^m \, \gamma^{(m)} \tag{3.12}$$

the last two equations read

$$\gamma_x \;=\; |\psi|^2 + i\,\lambda\,[\gamma_{xx} - (\bar{\psi}_x/\bar{\psi})\gamma_x] - \lambda^2(\gamma_x)^2 \tag{3.13}$$

$$\gamma_t \;=\; i\,[\gamma_{xx} - 2\,(\bar{\psi}_x/\bar{\psi})\,\gamma_x] - \lambda\,(\gamma_x)^2 \,. \tag{3.14}$$

Differentiation of the second equation with respect to x leads to the conservation law

$$\gamma_{xt} \;=\; [i\,\gamma_{xx} - 2\,i\,(\bar{\psi}_x/\bar{\psi})\,\gamma_x - \lambda\,(\gamma_x)^2]_x \tag{3.15}$$

for γ_x. Inserting the power series expansion for γ in (3.13), we obtain

$$\gamma_x^{(0)} = |\psi|^2 \,, \qquad \gamma_x^{(1)} = i\,\bar{\psi}\,\psi_x \tag{3.16}$$

and the recursion formula

$$\gamma_x^{(m)} = i\,[\gamma_{xx}^{(m-1)} - (\bar{\psi}_x/\bar{\psi})\,\gamma_x^{(m-1)}] - \sum_{k=0}^{m-2} \gamma_x^{(k)}\,\gamma_x^{(m-2-k)} \tag{3.17}$$

for $m > 1$. In particular, $\gamma_x^{(2)} = -|\psi|^4 - \bar{\psi}\,\psi_{xx}$. These are the well-known conserved quantities of the nonlinear Schrödinger equation (cf [8], p.38, for example). (3.16) and (3.17) solve (3.13). (3.14) becomes $\gamma_t^{(0)} = i\,[\gamma_{xx}^{(0)} - 2\,(\bar{\psi}_x/\bar{\psi})\,\gamma_x^{(0)}]$ and

$$\gamma_t^{(m)} = i\,[\gamma_{xx}^{(m)} - 2\,(\bar{\psi}_x/\bar{\psi})\,\gamma_x^{(m)}] - \sum_{k=0}^{m-1} \gamma_x^{(k)}\,\gamma_x^{(m-1-k)} \tag{3.18}$$

for $m > 0$. Inserting our solution $\gamma^{(m)} = \int^x \gamma_{x'}^{(m)}\,dx'$ with the $\gamma_x^{(m)}$ as given above, these equations are satisfied as a consequence of the nonlinear Schrödinger equation (which in turn is a consequence of the compatibility of (3.13) and (3.14), respectively (3.10) and (3.11)).

4 Nonlinear Schrödinger equation in noncommutative space-time

The passage from commutative to noncommutative space-time is (in the present context) achieved by replacing the ordinary commutative product in the space of smooth functions

on $\mathbb{R}^2$ with coordinates t, x by the noncommutative associative (Moyal) $*$-product [2] which is defined by

$$f * h = m \circ e^{\vartheta P/2}(f \otimes h) \tag{4.1}$$

where ϑ is a real or imaginary constant and

$$m(f \otimes h) = f\,h\,, \qquad P = \partial_t \otimes \partial_x - \partial_x \otimes \partial_t\ . \tag{4.2}$$

As a consequence, one finds[4]

$$f * h - h * f = 2\,m \circ \sinh(\vartheta P/2)(f \otimes h) = \vartheta\, m \circ \frac{\sinh(\vartheta P/2)}{\vartheta P/2} \circ P(f \otimes h)\ . \tag{4.3}$$

In the following, the product[5] defined by

$$f \diamond h = m \circ \frac{\sinh(\vartheta P/2)}{\vartheta P/2}(f \otimes h) \tag{4.4}$$

will be of some help for us.

In noncommutative space-time, the two bicomplex maps associated with the nonlinear Schrödinger equation should be replaced by

$$\mathrm{d}\phi = (\phi_t - V * \phi)\,\tau + (\phi_x - U * \phi)\,\xi \tag{4.5}$$

$$\delta\phi = \phi_x\,\tau + \frac{1}{2i}(I - \sigma_3)\phi\,\xi \tag{4.6}$$

with 2×2 matrices U and V. The bicomplex space M is the same as in the previous section, however. For the following calculations it is important to note that partial derivatives are also derivations with respect to the $*$-product. The bicomplex conditions imply

$$U_t - V_x + U * V - V * U = 0 \tag{4.7}$$

and

$$U_x = \frac{i}{2}\,(\sigma_3\,V - V\,\sigma_3)\ . \tag{4.8}$$

Substituting the decomposition

$$V = i\,(V^+ + V^-)\,\sigma_3, \qquad \text{with} \qquad \sigma_3\,V^\pm = \pm V^\pm\,\sigma_3 \tag{4.9}$$

in (4.8), we obtain

$$V^- = U_x\ . \tag{4.10}$$

[4]This is easily verified using $\tau(f \otimes h) = h \otimes f$ which has the properties $P \circ \tau = -\tau \circ P$ and $m \circ \tau = m$.
[5]This product has been previously used in [5].

This suggests to impose the condition

$$\sigma_3\, U = -U\, \sigma_3 \tag{4.11}$$

on U. Inserting this in (4.7), we obtain $V_x^+ = (U * U)_x$ and thus

$$V^+ = U * U \tag{4.12}$$

up to addition of an arbitrary term which does not depend on x and which we disregard in the following. Hence

$$V = i\,(U_x + U * U)\,\sigma_3\;. \tag{4.13}$$

Furthermore, (4.7) together with (4.12) leads to

$$i\,U_t\,\sigma_3 + U_{xx} - 2\,U * U * U = 0\;. \tag{4.14}$$

If we impose the further condition $U^\dagger = -U$, then

$$U = \begin{pmatrix} 0 & -\bar{\psi} \\ \psi & 0 \end{pmatrix} \qquad V = i\begin{pmatrix} -\bar{\psi} * \psi & \bar{\psi}_x \\ \psi_x & \psi * \bar{\psi} \end{pmatrix} \tag{4.15}$$

with a complex function ψ and (4.14) takes the form

$$i\,\psi_t + \psi_{xx} + 2\,\psi * \bar{\psi} * \psi = 0\,, \qquad i\,\bar{\psi}_t - \bar{\psi}_{xx} - 2\,\bar{\psi} * \psi * \bar{\psi} = 0 \tag{4.16}$$

which are the *noncommutative nonlinear Schrödinger equation* (NNS) and its complex conjugate.[6]

The linear system $\delta\chi = \lambda\,\mathrm{d}\chi$ (where $\delta\chi^{(0)} = 0$) associated with the NNS reads

$$\chi_x = \lambda\,(\chi_t - V * \chi)\,, \qquad (I - \sigma_3)\chi = 2\,i\,\lambda\,(\chi_x - U * \chi)\;. \tag{4.17}$$

Writing $\chi = (\alpha, \beta)^T$, this becomes

$$\begin{aligned} \alpha_x &= \lambda\,(\alpha_t + i\,\bar{\psi} * \psi * \alpha - i\,\bar{\psi}_x * \beta) && (4.18)\\ \beta_x &= \lambda\,(\beta_t - i\,\psi_x * \alpha - i\,\psi * \bar{\psi} * \beta) && (4.19)\\ 0 &= \lambda\,(\alpha_x + \bar{\psi} * \beta) && (4.20)\\ \beta &= i\,\lambda\,(\beta_x - \psi * \alpha)\;. && (4.21) \end{aligned}$$

Assuming that ψ is $*$-invertible with inverse ψ_*^{-1}, we can use the third equation to eliminate β from the first and the last equation,

$$\begin{aligned} \alpha_x &= \lambda\,(\alpha_t + i\,\bar{\psi} * \psi * \alpha + i\,\bar{\psi}_x * \bar{\psi}_*^{-1} * \alpha_x) && (4.22)\\ \alpha_x &= i\,\lambda\,(\alpha_{xx} - \bar{\psi}_x * \bar{\psi}_*^{-1} * \alpha_x + \bar{\psi} * \psi * \alpha)\;. && (4.23) \end{aligned}$$

[6] The two equations are complex conjugate irrespective of whether ϑ is real or imaginary. Clearly $\overline{f * h} = \bar{f} * \bar{h}$ if ϑ is real. If ϑ is imaginary, we have $\overline{f * h} = \bar{h} * \bar{f}$ instead.

It is not possible now to proceed in perfect analogy with (3.12) of the classical case. Instead, we introduce functions p and q such that

$$\alpha_t = i\,\lambda\, p * \alpha\,, \qquad \alpha_x = i\,\lambda\, q * \alpha \tag{4.24}$$

assuming that α is $*$-invertible. In terms of these functions the above equations take the form

$$\begin{aligned} q &= \bar{\psi} * \psi + i\,\lambda\,(q_x - \bar{\psi}_x * \bar{\psi}_*^{-1} * q) - \lambda^2\, q * q \qquad (4.25)\\ p &= i\, q_x - 2\, i\, \bar{\psi}_x * \bar{\psi}_*^{-1} * q - \lambda\, q * q\,. \qquad (4.26)\end{aligned}$$

From $\alpha_{tx} = \alpha_{xt}$ and (4.24) we find

$$q_t - p_x + i\,\lambda\,(q * p - p * q) = 0\,. \tag{4.27}$$

Using (4.3) and (4.4), we find

$$q * p - p * q = \vartheta\,(q_t \diamond p_x - q_x \diamond p_t) = \vartheta\,(q \diamond p_x)_t - \vartheta\,(q \diamond p_t)_x \tag{4.28}$$

where, in the last step, we used the fact that partial derivatives are also derivations with respect to the $\diamond$-product.[7] Now we insert this in (4.27) and deduce

$$w_t = (p + i\,\lambda\,\vartheta\, q \diamond p_t)_x \tag{4.29}$$

where

$$w = q + i\,\lambda\,\vartheta\, q \diamond p_x\,. \tag{4.30}$$

By expansion into a power series in λ, this yields an infinite set of local conservation laws of the NNS (each of which, in turn, can be expanded into a power series in ϑ). Let us expand p and q into power series in λ,

$$p = \sum_{m=0}^{\infty} \lambda^m\, p^{(m)}\,, \qquad q = \sum_{m=0}^{\infty} \lambda^m\, q^{(m)}\,. \tag{4.31}$$

Then (4.25) leads to

$$q^{(0)} = \bar{\psi} * \psi\,, \qquad q^{(1)} = i\,\bar{\psi} * \psi_x \tag{4.32}$$

and

$$q^{(m)} = i\,(q_x^{(m-1)} - \bar{\psi}_x * \bar{\psi}_*^{-1} * q^{(m-1)}) - \sum_{k=0}^{m-2} q^{(k)} * q^{(m-2-k)} \tag{4.33}$$

[7]Note that, for a partial derivative ∂, we have $\partial \circ m = m \circ \partial_\otimes$ with $\partial_\otimes = \partial \otimes 1 + 1 \otimes \partial$ which commutes with P.

for $m > 1$. From (4.26) we get

$$p^{(0)} = i\,(\bar{\psi} * \psi_x - \bar{\psi}_x * \psi) \tag{4.34}$$

and

$$p^{(m)} = i\,(q_x^{(m)} - 2\,\bar{\psi}_x * \bar{\psi}_*^{-1} * q^{(m)}) - \sum_{k=0}^{m-1} q^{(k)} * q^{(m-1-k)} \tag{4.35}$$

for $m > 0$. These formulas allow the recursive calculation of the functions $p^{(m)}$ and $q^{(m)}$ in terms of ψ. From (4.30) with $w = \sum_{m\geq 0} w^{(m)}$ we now obtain the following expressions for the conserved densities,

$$\begin{aligned} w^{(0)} &= \bar{\psi} * \psi & (4.36)\\ w^{(1)} &= i\,\bar{\psi} * \psi_x - \vartheta\,(\bar{\psi} * \psi) \diamond (\bar{\psi} * \psi_{xx} - \bar{\psi}_{xx} * \psi) & (4.37)\\ w^{(m)} &= q^{(m)} + i\,\vartheta \sum_{k=0}^{m-1} q^{(k)} \diamond p_x^{(m-1-k)} \qquad (m > 1)\,. & (4.38) \end{aligned}$$

5 Single soliton solution of the NNS

The one-soliton solution of the classical nonlinear Schrödinger equation is given by

$$\psi = a\,\exp\left(\frac{i}{2}\,b\,x - i\,(\frac{1}{4}\,b^2 - a^2)\,t\right)\,\mathrm{sech}[a\,(x - b\,t)] \tag{5.1}$$

with real constants a, b. This expression can be rewritten as follows,

$$\psi = a\,F(x) * G(x,t) * F(x) \tag{5.2}$$

where

$$F(x) = \exp\left(\frac{i}{2}\,(\frac{b}{4} + \frac{a^2}{b})\,x\right)\,, \quad G(x,t) = \exp\left(i\,(\frac{b}{4} - \frac{a^2}{b})(x - b\,t)\right)\,\mathrm{sech}[a\,(x - b\,t)]\,. \tag{5.3}$$

Clearly, this reduces to the classical solution if $\vartheta = 0$. It is therefore sufficient to show that (5.2) does not depend on ϑ. We make use of the identity

$$\frac{\partial}{\partial\vartheta}(f * h) = \frac{\partial f}{\partial\vartheta} * h + f * \frac{\partial h}{\partial\vartheta} + \frac{1}{2}(f_t * h_x - f_x * h_t) \tag{5.4}$$

for functions f and h. Applying this to functions $f(x), g(x,t), h(x)$ which do not depend on ϑ, we find

$$\frac{\partial}{\partial\vartheta}(f * g * h) = \frac{1}{2}(f * g_t * h_x - f_x * g_t * h) \tag{5.5}$$

which vanishes if $f_x = c\,f$, $h_x = c\,h$ with a constant c (for arbitrary g). Since (5.2) has this special form, it is indeed independent of ϑ.

Our next observation is that a $*$-product of two functions is classical if the functions depend both on the same argument linear in x and t and if they do not depend on ϑ. Hence $f(x-b\,t) * h(x-b\,t) = f(x-b\,t)\,h(x-b\,t)$. Indeed, by application of (5.4) we have

$$\frac{\partial}{\partial\vartheta}[f(x-b\,t) * h(x-b\,t)] = \frac{1}{2}(-b\,f' * h' + b\,f' * h') = 0 \tag{5.6}$$

where f' denotes the derivative of f. As a consequence,

$$\begin{aligned} \psi * \bar\psi * \psi &= a^3\,F(x) * G(x,t) * \overline{G(x,t)} * G(x,t) * F(x) \\ &= a^3\,F(x) * \left[\exp\left(i\,(\frac{b}{4} - \frac{a^2}{b})(x-b\,t)\right)\,\mathrm{sech}^3[a\,(x-b\,t)]\right] * F(x) \\ &= |\psi|^2\,\psi \end{aligned} \tag{5.7}$$

using again our previous argument in the last step. Therefore the NNS reduces to the classical equation in case of the single soliton solution. We have thus shown that the single soliton solution of the classical nonlinear Schrödinger equation remains a solution of the NNS. A similar result should not be expected for multi-soliton solutions.

6 Some perturbative properties of the NNS

Expansion of (4.1) leads to

$$f * h = f\,h + \frac{\vartheta}{2}\,(f_t\,h_x - f_x\,h_t) + \frac{\vartheta^2}{8}\,(f_{tt}\,h_{xx} - 2\,f_{tx}\,h_{tx} + f_{xx}\,h_{tt}) + \mathcal{O}(\vartheta^3)\;. \tag{6.1}$$

Since f and h in general depend on ϑ, corresponding power series expansions have to be inserted in the last expression and terms rearranged in ascending powers of ϑ. Using this formula, we obtain the following expression for the nonlinear term in the NNS,

$$\begin{aligned} \psi * \bar\psi * \psi &= |\psi|^2\,\psi + (\vartheta^2/4)\,\big([\psi_{tt}\psi_{xx} - (\psi_{tx})^2]\,\bar\psi + 2\,[\psi_t\psi_{xx} - \psi_x\psi_{tx}]\,\bar\psi_t \\ &\quad +2\,[\psi_{tt}\psi_x - \psi_{tx}\psi_t]\,\bar\psi_x + [\psi\psi_{xx} - (\psi_x)^2]\,\bar\psi_{tt} + [\psi\psi_{tt} - (\psi_t)^2]\,\bar\psi_{xx} \\ &\quad +2\,[\psi_t\psi_x - \psi\psi_{tx}]\,\bar\psi_{tx}\big) + \mathcal{O}(\vartheta^3)\;. \end{aligned} \tag{6.2}$$

In particular, the contribution of first order in ϑ from the expansion of the $*$-product vanishes identically. By expansion of ψ, i.e.,

$$\psi = \psi_0 + \vartheta\,\psi_1 + \frac{1}{2}\,\vartheta^2\,\psi_2 + \mathcal{O}(\vartheta^3)\,, \tag{6.3}$$

terms linear in ϑ arise, however. Now the first correction to the nonlinear Schrödinger equation obtained from the NNS is given by

$$i\,\psi_{1,t} + \psi_{1,xx} + 4\,|\psi_0|^2\,\psi_1 \pm 2\,\psi_0^2\,\bar\psi_1 = 0 \tag{6.4}$$

where the choice of sign depends on whether ϑ is chosen real or imaginary. This equation is linear and homogeneous in ψ_1 and thus admits the solution $\psi_1 = 0$. As a consequence, every solution of the classical nonlinear Schrödinger equation is a solution of the NNS to first order in ϑ. The second correction (quadratic in ϑ) to the nonlinear Schrödinger equation is an inhomogenous linear equation for ψ_2,

$$i\,\psi_{2,t} + \psi_{2,xx} + 2\,|\psi_0|^2\,\psi_2 + \psi_0^2\,\bar{\psi}_2 = -2\,\psi_1^2\,\bar{\psi}_0 \mp 4\,\psi_0\,|\psi_1|^2 - \Gamma/2 \tag{6.5}$$

with

$$\begin{aligned}
\Gamma \;=\; & -4\,\psi_0^4\,\bar{\psi}_0\,\bar{\psi}_{0,x}^2 + 3\,\psi_{0,xx}^2\,\bar{\psi}_{0,xx} - 2\,\psi_{0,x}\,\bar{\psi}_{0,xx}\,\psi_{0,xxx} + \bar{\psi}_0\,\psi_{0,xxx}^2 \\
& -4\,\psi_0^2\,[\bar{\psi}_0^3\,\psi_{0,x}^2 + 3\,\psi_{0,x}\,\bar{\psi}_{0,x}\,\bar{\psi}_{0,xx} + \bar{\psi}_0\,(2\,\psi_{0,xx}\,\bar{\psi}_{0,xx} + \psi_{0,x}\,\bar{\psi}_{0,xxx})] \\
& -4\,\psi_0^3\,\bar{\psi}_{0,x}\,(2\,\bar{\psi}_0^2\,\psi_{0,x} + \bar{\psi}_{0,xxx}) + \psi_{0,xx}\,(2\,\bar{\psi}_{0,x}\,\psi_{0,xxx} + 2\,\psi_{0,x}\,\bar{\psi}_{0,xxx} - \bar{\psi}_0\,\psi_{0,xxxx}) \\
& -2\,\psi_{0,x}\,\bar{\psi}_{0,x}\,\psi_{0,xxxx} + \psi_{0,x}^2\,\bar{\psi}_{0,xxxx} - \psi_0\,[16\,\bar{\psi}_0\,\psi_{0,x}\,\bar{\psi}_{0,x}\,\psi_{0,xx} + 4\,\bar{\psi}_0^2\,\psi_{0,xx}^2 \\
& +4\,\psi_{0,x}^2\,(3\,\bar{\psi}_{0,x}^2 + \bar{\psi}_0\,\bar{\psi}_{0,xx}) + 2\,\psi_{0,xxx}\,\bar{\psi}_{0,xxx} + \bar{\psi}_{0,xx}\,\psi_{0,xxxx} + \psi_{0,xx}\,\bar{\psi}_{0,xxxx}]
\end{aligned} \tag{6.6}$$

where we used that ψ_0 satisfies the classical nonlinear Schrödinger equation.

Expanding the expressions for the conserved densities $w^{(0)}$ and $w^{(1)}$ (see (4.36) and (4.37)) in powers of ϑ, we obtain

$$\begin{aligned}
w^{(0)} &= \bar{\psi}_0\,\psi_0 + \vartheta\,[\bar{\psi}_0\psi_1 + \bar{\psi}_1\psi_0 + \frac{1}{2}\,(\bar{\psi}_{0,t}\,\psi_{0,x} - \bar{\psi}_{0,x}\,\psi_{0,t})] + \mathcal{O}(\vartheta^2) \\
&= \bar{\psi}_0\,\psi_0 + \vartheta\,[\bar{\psi}_0\psi_1 + \bar{\psi}_1\psi_0 - \frac{i}{2}\,(\bar{\psi}_{0,x}\,\psi_{0,x} + |\psi_0|^4)_x] + \mathcal{O}(\vartheta^2)
\end{aligned} \tag{6.7}$$

$$\begin{aligned}
w^{(1)} &= i\,\bar{\psi}_0\,\psi_{0,x} + \vartheta\,[i\,(\bar{\psi}_0\,\psi_{1,x} + \bar{\psi}_1\,\psi_{0,x}) \\
&\quad + \frac{i}{2}\,(\bar{\psi}_{0,t}\,\psi_{0,xx} - \bar{\psi}_{0,x}\,\psi_{0,xt}) - |\psi_0|^2\,(\bar{\psi}_0\,\psi_{0,xx} - \bar{\psi}_{0,xx}\,\psi_0)] + \mathcal{O}(\vartheta^2) \\
&= i\,\bar{\psi}_0\,\psi_{0,x} + \vartheta\,[i\,(\bar{\psi}_0\,\psi_{1,x} + \bar{\psi}_1\,\psi_{0,x}) + \frac{i}{2}\,(\bar{\psi}_{0,x}\,\psi_{0,xx} + \bar{\psi}_0\,\bar{\psi}_{0,x}\,\psi_0^2)_x] + \mathcal{O}(\vartheta^2)\;.
\end{aligned} \tag{6.8}$$

In the last step we used again that ψ_0 solves the classical nonlinear Schrödinger equation. Hence, those parts of the first order corrections to the conserved densities which do not depend on ψ_1 are total x-derivatives and thus do not contribute to the conserved charges. A corresponding evaluation of the second order corrections is much more complicated since one has to solve the inhomogeneous equation (6.5).

7 Conclusions

By deformation quantization applied to a bicomplex associated with the nonlinear Schrödinger equation, we obtained a deformed equation (NNS) which lives on a noncommutative space-time and which shares with its classical version the property of having an infinite set of conserved quantities, a characteristic feature of soliton equations and (infinite-dimensional) integrable models. Surprisingly, the one-soliton solution of the classical equation remains a

solution of the NNS. The fate of the classical multi-soliton structure under the deformation has still to be explored.

We expect that in a similar way interesting noncommutative versions of many other integrable models can be constructed (see also [9]).

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